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AN EFFICIENT BINOMIAL METHOD FOR PRICING ASIAN OPTIONS

Abstract. *We construct an efficient tree method for pricing path-dependent Asian options. The standard tree method estimates option prices at each node of the tree, while the proposed method defines an interval about each node along the stock price axis and estimates the average option price over each interval. The proposed method can be used independently to construct a new tree method, or it can be combined with other existing tree methods to improve the accuracy. Numerical results show that the proposed schemes show superiority in accuracy to other tree methods when applied to discrete forward-starting Asian options and continuous European or American Asian options.*

Keywords *: binomial tree method, cell averaging, Asian options.*

JEL Classification: G13, C63, C02

1. Introduction

An option is a financial derivative which gives the owner the right, but not the obligation, to buy or sell an underlying asset for a given price on or before the expiration date. From the seminal papers of Black and Scholes (1973) and Merton (1973), the trading volume of options has been increased and exotic options with nonstandard payoff patterns have become more common in the over-the-counter market. Among them, an option with the payoff determined by the average underlying price over some pre-defined period of time is called an Asian option. An Asian option has been popular since it could reduce the risk of market manipulation of the underlying asset at maturity and the volatility inherent in the option. However, these Asian options based on arithmetic averages cannot be priced in a closed-form, and one needs to rely on its numerical approximation instead.

There have been many approaches to approximate the value of exotic options, such as binomial tree method, Monte Carlo simulation, finite difference method for solving Black-Scholes partial differential equations etc. Both the Monte Carlo method and finite difference method suffer from the difficulty to deal with early exercise without bias, whereas the binomial tree method by Cox, Ross and

Rubinstein (1979) are very popular due to its ease of implementation and simple extension to American type options. However, due to the averaging nature of Asian options, the number of averaging nodes in binomial tree grows exponentially. Therefore straightforward extension of the standard binomial method to Asian case is not possible in practice. In order to solve this shortage of binomial method, Hull and White (1993) considered a set of representative averages at each node including minimum and maximum average values. Employing this set of representative averages makes the binomial model feasible for pricing Asian options, though it still suffers the lack of convergence, see Costabile, Massabo and Russo (2006) and Forsyth, Vetzal and Zvan (2002). For a discrete monitored Asian option, Hsu and Lyuu (2011) proposed a quadratic-time convergent binomial method based on the Lagrange multiplier to choose the number of states for each node of a tree.

In this paper, based on the cell averaging approach in Moon and Kim (2013), small bins on the asset price axis, called cells, are defined about each node of the tree and then average option price over each cell has been computed and updated in time. See Section 2 for details. The binomial method of Hsu and Lyuu (2011) for discrete monitored Asian options and that of Hull and White (1993) for continuous Asian options have been modified for improvement with the help of cell averaging method. Numerical experiments in section 4 show that the proposed cell averaging binomial method gives more accurate results compared to other existing computational methods.

The outline of the paper is as follows. In section 2 we explain the problem and introduce the cell averaging binomial method. In section 3 we extend the cell averaging binomial method to discrete and continuous monitored Asian options. In section 4 we compare the accuracy and efficiency of the existing tree methods with those of the cell averaging binomial method. We finally summarize our conclusions in section 5.

2. Cell Averaging Binomial Methods

Let us consider the price of the underlying asset as a stochastic process $\{S(t), t \in [0,T]\}$ which satisfies the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad 0 < t < T, \tag{1}$$

where μ is an expected rate of return, σ is a volatility, T is an expiration date, and $W(t)$ is a Brownian motion. From the Ito formula in Øksendal (1998), $X(t) \equiv \ln(S(t))$ satisfies

$$dX(t) = (\mu - \sigma^2/2)dt + \sigma dW(t), \quad t > 0.$$

In the risk-neutral world, the value of the European option can be computed by the discounted conditional expectation of the terminal payoff,

$$V(x,t) = e^{-r(T-t)} E\left[\Lambda(X(T)) \mid X(t) = x\right],$$

where $\Lambda(X(T))$ is the payoff at $t = T$. Without loss of generality, we denote again the risk neutral process to be $X(t)$ with drift rate equal to the risk-free interest rate r, instead of μ in (1). If we consider a continuous dividend yield q, the drift rate becomes $r - q$.

2.1. The binomial model

Let us first discretize the time period $[0,T]$ into N intervals of the same length $\Delta t = T/N$, $0 = t_0 < t_1 < ... < t_N = T$. The standard binomial method by Cox, Ross and Rubinstein (1979) assumes that the asset price $S(t_n)$ at $t = t_n$ moves either up to $S(t_n)\,u$ for $u = \exp(\sigma\sqrt{\Delta t}) > 1$ or down to $S(t_n)\,d$ for $d = 1/u$ and $n = 0,\ 1,\ \ldots,\ N-1$ with probabilities $p = (e^{r\Delta t} - d)/(u-d)$ or $1-p$, respectively, or $X(t_n)$ in log at $t = t_n$ moves either up to $X(t_n)+h$ or down to $X(t_n)-h$ where $h = \ln u$. Let $X_j^n = X_0 + (2j-n)h$ denote the values at $t = t_n = n\Delta t$ for $j = 0,\ 1,\ \ldots,\ n$, with $X(0) = X_0$. Then the standard binomial method calculates the payoffs of the option at expiry, $V_j^N = \Lambda(X_j^N)$ for $j = 0,1,...,N$, and computes the option price $V_0^0 = V(X_0, 0)$ by backward averaging,

$$V(x,t) = e^{-r\Delta t}(pV(x+h,\ t+\Delta t)\ +\ (1-p)V(x-h,\ t+\Delta t)), \tag{2}$$

where $x = X_j^n$, $j = 0,\ \ldots,\ n$, and $n = N-1,\ N-2,\ \ldots,\ 0$.

The binomial method approximation converges to the Black-Scholes value as the number of time steps, N, tends to infinity, see the general theory in Kwok (1998), Clewlow and Strickland (1998), Lyuu (2002) and Higham (2004). However, it is widely reported that the convergence is not monotone and the saw-tooth pattern in the sequence of approximations makes the binomial approximation less attractive.

2.2 The cell averaging binomial model

In order to reduce the saw-tooth patterns in the sequence of approximations in the standard binomial method, we employ the cell averaging method. Let us first divide the interval $\left[X_0-(N+1)h,\ X_0+(N+1)h\right]$ on the X-axis into $N+1$ non-overlapping equidistant intervals of length $2h$, called cells, centered at points $X_0 + (2j-N)h$, $j = 0,\ \ldots,\ N$, then compute average option payoffs on each cell centered at X_j^n at expiry $t = t_N$,

$$\bar{V}_j^N \equiv \frac{1}{2h}\int_{X_j^N - h}^{X_j^N + h} \Lambda(\xi)\, d\xi, \quad j = 0, \ldots, N \tag{3}$$

where $\Lambda(\cdot)$ is the payoff function at expiry. If (2) is satisfied at every point $\xi \in \left[X_j^n - h,\ X_j^n + h\right]$ in the cell at time t_n, then the average option price $\bar{V}_j^n = \frac{1}{2h}\int_{X_j^n-h}^{X_j^n+h} V(\xi, t_n)\mathrm{d}\xi$ satisfies the following backward averaging relation

$$\bar{V}_j^n = e^{-r\Delta t}\left(p\bar{V}_{j+1}^{n+1} + (1-p)\bar{V}_j^{n+1}\right) \tag{4}$$

See Figure 1. Appropriate modification will be needed if (2) does not hold at every point $\xi \in \left[X_j^n - h,\ X_j^n + h \right]$. For instance, see Moon and Kim (2013) for the case of barrier options. Then cell averages of the option values at expiry (3) can be updated iteratively, which eventually leads to the average of the option price $\bar{V}_0^0$ on $\left[X_0 - h,\ X_0 + h \right]$ at $t = 0$.

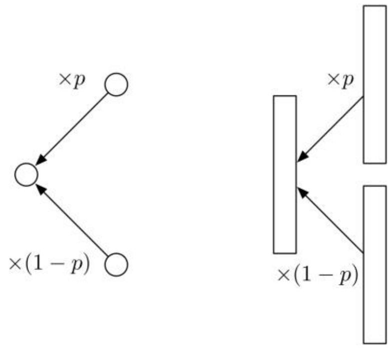

Figure 1 : Comparison of backward averaging between the standard tree method (Left) and the cell averaging tree method (Right).

Figure 2 compares the standard binomial method (dash) and the cell averaging binomial method (solid) for the European up-and-out barrier put option price. The figure shows that this cell averaging idea reduces jig-saw oscillations.

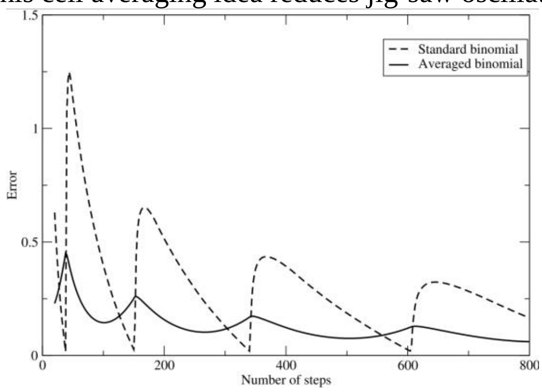

Figure 2 : Parameters: The initial stock price $S(0) = 100$, the risk-free interest rate $r = 0.05$, the volatility $\sigma = 0.3$, strike price $K = 90$, barrier $H = 105$ and the maturity $T = 1$ for a European up-and-out barrier put option. Comparison between standard binomial (dash) and cell averaging binomial (solid) lattice models. Cell averaging produces smoother convergence.

3. Asian Options

Now we extend the cell averaging binomial method in section 2 to path-dependent Asian options. As it has been known, there do not exist explicit closed-form analytical solutions for arithmetic Asian options because the arithmetic average of a set of lognormal random variables is not log-normally distributed. For that reason, many numerical approaches have been proposed. We first consider discrete monitored Asian options in Section 3.1 and modify the method of Hsu and Lyuu (2011) to price it. Then we improve the method of Hull and White (1993) in

Section 3.2 to price the continuous Asian option.

3.1 The Discrete Monitored Asian option

The discretely monitored Asian option is often found in practice. The payoff of Asian call option with strike price K at the expiry date T is given by the following :

$$\max\left(\frac{1}{n}\sum_{i=1}^{n} S(t_i) - K, 0\right) \tag{5}$$

where n is the number of monitor points and the payoff of discrete type arithmetic average Asian call option is monitored at n time points, $0 \le t_1 < t_2 < \cdots < t_n \le T$. We assume each time interval between two adjacent monitor points is partitioned into I time steps, and I is called intraday. Then we see that the monitor points are at times $0, I, 2I, \ldots, nI$ and the whole number of time steps is $N = nI$. For the standard European-style discrete Asian call, the payoff at expiry is

$$\max\left(\frac{1}{n+1}\sum_{i=0}^{n} S_{iI} - K, 0\right)$$

whereas the *forward-starting* discrete Asian option omits the initial S_0 and has the payoff

$$\max\left(\frac{1}{n}\sum_{i=1}^{n} S_{iI} - K, 0\right)$$

In order to be self-contained, we start with explanation of the binomial method by Hsu and Lyuu (2011) which follows the standard binomial method suggested by Cox, Ross and Rubinstein (1979). Let $N(i, j)$ denote the node at time i that results from j down moves and $i - j$ up moves. Then the price sum to expiry date for a price path $(S_0, S_1, \ldots, S_i)$, P, called the running sum, is computed by

$$P \equiv \begin{cases} S_0 + S_I + \cdots + S_{\lfloor i/I \rfloor \cdot I} & \text{for standard Asian options} \\ S_I + S_{2I} + \cdots + S_{\lfloor i/I \rfloor \cdot I} & \text{for forward-starting Asian options,} \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes floor function and $0 \le i \le N$. Since the pricing of Asian option using binomial lattice produces 2^N possible paths for each time step N, Hsu and Lyuu (2011) suggested a discrete binomial method for Asian option pricing, where they proposed the running sum P of the form

$$P = \begin{cases} 0, \dfrac{(n+1)K}{k_{ij}}, \dfrac{2(n+1)K}{k_{ij}}, \cdots, \dfrac{(k_{ij}-1)(n+1)K}{k_{ij}}, (n+1)K & \text{if } i \ne 0 \\ S_0 & \text{if } i = 0 \end{cases} \tag{6}$$

for standard Asian option

$$P = \begin{cases} 0, \dfrac{nK}{k_{ij}}, \dfrac{2nK}{k_{ij}}, \cdots, \dfrac{(k_{ij}-1)nK}{k_{ij}}, nK & \text{if } i \neq 0 \\ 0 & \text{if } i = 0 \end{cases} \tag{7}$$

for forward-staring Asian option where $k_{ij}+1$ represents the number of states considered for each node $N(iI, j)$. Here k_{ij} is computed by

$$k_{ij} = \frac{cIn^2}{2} \frac{\left(\dfrac{B(iI, j, p)}{i^2}\right)^{\frac{1}{3}}}{\displaystyle\sum_{s=1}^{n}\sum_{t=0}^{sI}\left(\dfrac{B(sI, t, p)}{s^2}\right)^{\frac{1}{3}}}, \tag{8}$$

where $B(i, j, p) = \begin{pmatrix} i \\ j \end{pmatrix} p^{i-j}(1-p)^j$, and c is the average number of states per node. If the 3-tuple (iI, S, P) denotes the current state, the corresponding option value $V(iI, S, P_{iI})$ can be computed by

$$V(iI, S, P_{iI}) = \frac{1}{R^I}\sum_{l=0}^{I} p_l \; V((i+1)I, \; Su^{I-2l}, \; P_{iI} + Su^{I-2l}),$$

where $S = S_0 u^{iI-2j}$, $R = \exp(r\Delta t)$, and the associated branching probabilities are

$$p_l \equiv \begin{pmatrix} I \\ l \end{pmatrix} p^{I-l} \; (1-p)^l$$

for each branch $l = 0, \ldots, I$, and $i = 0, 1, \ldots, (n-1)$. When $P_{iI} > (n+1)K$,

$$V(iI, S, P_{iI}) = \begin{cases} \{P_{iI} + (n-i)S\}/(n+1) - K & \text{if } R = 1 \\ R^{-(n-i)I}\{(P_{iI} + SR^I \dfrac{1-R^{(n-i)I}}{1-R^I})/(n+1) - K\} & \text{if } R > 1. \end{cases}$$

For forward-starting discrete Asian options, the similar formulas hold

$$V(iI, S, P_{iI}) = \begin{cases} \{P_{iI} + (n-i)S\}/n - K & \text{if } R = 1 \\ R^{-(n-i)I}\{(P_{iI} + SR^I \dfrac{1-R^{(n-i)I}}{1-R^I})/n - K\} & \text{if } R > 1 \end{cases}$$

when $P_{iI} > nK$. Otherwise, linear interpolation is computed from the two bracketing running sums' corresponding option values to obtain :

$$V((i+1)I, Su^{I-2l}, P_{iI} + Su^{I-2l}) =$$

$$\alpha_l V\left((i+1)I, Su^{I-2l}, (s_l - 1)\frac{(n+1)K}{k_{i+1, j+l}}\right) + (1-\alpha_l)V\left((i+1)I, Su^{I-2l}, s_l \frac{(n+1)K}{k_{i+1, j+l}}\right),$$

where $0 \le \alpha_l \le 1$ for $l = 0,1,\dots,I$.

Now let us apply cell averaging algorithm to this discrete model over the cells on the X-axis as in Section 2.2. For example, the cell-averaged payoffs at expiry for standard discrete Asian call option are computed as follows:

$$\frac{1}{2h}\int_{x^*-h}^{x^*+h} \max\left(\frac{1}{n+1}\left(P+e^x\right)-K,0\right)\,\mathrm{d}x,$$

where $x^* = \ln Su^{I-2l}$ for S, the stock price at $t=(n-1)I\Delta t$. Algorithm 1 shows the cell averaging tree algorithm for pricing the European standard discrete Asian call option based on method of Hsu and Lyuu (2011) method.

Algorithm 1 (HL-CA) Pricing European standard discrete Asian call option using Hsu and Lyuu model with cell averaging method

Require: $S_0, K, r, \sigma, \tau, n, I, c, u, d, R$;
$\Delta t := \tau/(nI)$; $p := (R-d)/(u-d)$; $h := \ln u$;
for $j = 0 \to (n-1)I$ **do** ▷ Calculate payoff for last step
  Let $P_{n-1,j}[\cdot]$ be running sums for $N(n-1,j)$;
  Let $S = S_0 u^{(n-1)I-2j}$;
  for $l = 0 \to I$ **do**
    Set $x^* = \ln(Su^{I-2l})$;
    $\Lambda[l][\cdot] := \frac{1}{2h}\int_{x^*-h}^{x^*+h}\max(\frac{1}{n+1}(P_{n-1,j}[\cdot]+e^x)-K,0)\ \mathrm{d}x$
  end for
  $C[j][\cdot] := R^{-I}\sum_{l=0}^{I}\binom{I}{l}p^{I-l}(1-p)^l\Lambda[l][\cdot]$
end for
for $i = (n-2) \to 0$ **do** ▷ Update option value using backward induction
  for $j = 0 \to iI$ **do**
    Let $P_{i,j}[\cdot]$ be running sums for $N(i,j)$;
    Set $S = S_0 u^{iI-2j}$;
    for $l = 0 \to I$ **do**
      if $(P_{i,j}[\cdot] + Su^{I-2l}) > (n+1)K$ **then**
        $V[j][\cdot] := R^{-(n-i-1)I}\left(\left(P_{i,j}[\cdot] + Su^{I-2l}\frac{1-R^{(n-i)I}}{1-R^I}\right)/(n+1) - K\right)$
      else
        Let α_l be such that
        $P_{i,j}[\cdot] + Su^{I-2l} = \alpha_l P_{i+1,j+l}[s_l-1] + (1-\alpha_l)P_{i+1,j+l}[s_l]$
        for some $0 \le s_l \le k_{i+1,j+l}$;
        $V[j][\cdot] := \alpha_l C[j+l][s_l-1] + (1-\alpha_l)C[j+l][s_l]$
      end if
    end for
    $C[j][\cdot] := R^{-I}\sum_{l=0}^{I}\binom{I}{l}p^{I-l}(1-p)^l V[j][\cdot]$
  end for
end for
$C[1][1]$; ▷ C[1][1] is the option price

3.2. The Continuous Asian option

Now we consider the case of a continuously monitored Asian option. The payoff for continuously monitored Asian call option with strike price K at the expiry date T is given by the following :

$$\max\left(\frac{1}{T}\int_0^T S(\tau)\,d\tau - K\right) \tag{9}$$

Let us consider Hull and White (1993) binomial model. As explained above, pricing of Asian option using binomial tree produces 2^N possible paths for each time step N. Hull and White (1993) proposed a binomial model for continuous Asian options to solve this problem. They chose the representative average values of the form $S_0 e^{\pm mh}$ for each node, where h is a fixed constant, and $S_0 = S_{0,0}$ is the known initial asset price. Let A_i be the representative averages, and let A_i^{min} and A_i^{max} be the minimum and maximum representative average, respectively at time $i\Delta t$, $i = 1, 2, \cdots, n$. Then m is the smallest integer chosen to satisfy by following inequalities :

$$A_i^{min} = S_0 e^{-mh} \le \frac{1}{i+1}\left(iA_{i-1}^{min} + dS_{i-1,0}\right),$$

$$A_i^{max} = S_0 e^{mh} \ge \frac{1}{i+1}\left(iA_{i-1}^{max} + uS_{i-1,i-1}\right),$$

See Costabile, Massabo and Russo (2006) for details. We can also compute the averages of the form $S_0 e^{kh}$, where $k = -m+1, -m+2, \ldots, m-1$ for each time step. Hull and White model also follows the standard lattice binomial method proposed by Cox, Ross and Rubinstein (1979) and use the backward induction procedure

$$V(i, j, k) = e^{-r\Delta t}\left[pV(i+1, j+1, k_u) + qV(i+1, j, k_d)\right].$$

$V(i+1, j+1, k_u)$ is generated by using linear interpolation as follow. First, $\left[(i+1)A_i + uS_{i,j}\right]/(i+2)$ is computed and let A_k be the smallest representative average greater than $\left[(i+1)A_i + uS_{i,j}\right]/(i+2)$. Then $V(i+1, j+1, k_u)$ is the interpolation between two option values associated to A_{k-1} and A_k. The value $V(i+1, j+1, k_d)$ is derived in a similar way.

Now, let us modify Hull and White model for improvement by using cell averages in Section 2.2. For instance, we replace the payoff in of Hull and White model $\max(A_n - K, 0)$ for the representative averages A_n at the last time step with

$$\frac{1}{2\alpha}\int_{S_0 e^{kh}-\alpha}^{S_0 e^{kh}+\alpha} \max(x - K, 0) \tag{10}$$

where α is fixed constant and k represents all integers between $-m$ and m. Algorithm 2 shows the cell averaging tree algorithm for pricing the European continuous Asian call option based on method of Hull and White (1993).

Algorithm 2 (HW-CA) Pricing European continuous Asian call option using Hull and White binomial model with cell averaging

Require: $S_0, K, r, \sigma, T, n, h, u, d$;

$\Delta t := T/n$; $p := (e^{r\Delta t} - d)/(u - d)$;
for $i = 0 \to n$ **do** ▷ Find m for each time step
  Find the smallest m such that simultaneously satisfies
  $A_i^{min} = S_0 e^{-mh} \leq \frac{1}{i+1}\left(iA_{i-1}^{min} + dS_{i-1,0}\right)$,
  $A_i^{max} = S_0 e^{mh} \geq \frac{1}{i+1}\left(iA_{i-1}^{max} + uS_{i-1,i-1}\right)$.
  $M[i] := m$;
end for
for $i = 0 \to n$ **do** ▷ Calculate payoff for last step
  for $j = -M[n] \to M[n]$ **do**
    $C[i][j] := \frac{1}{2\alpha}\int_{S_0 e^{jh}-\alpha}^{S_0 e^{jh}+\alpha} \max(x - K, 0)\mathrm{d}x$;
  end for
end for
for $i = n - 1 \to 0$ **do** ▷ Update value using backward induction
  for $j = 0 \to i$ **do**
    Set $S = S_0 u^i d^j$;
    for $s = -M[i] \to M[i]$ **do**
      $Avg_u := (iA_i[s] + uS)/(i + 1)$;
      Let l be such that $A_i[l] \leq Avg_u \leq A_i[l + 1]$;
      Let α be such that $Avg_u = \alpha A_i[l] + (1 - \alpha)A_i[l + 1]$;
      $C_u := \alpha C[j][l] + (1 - \alpha)C[j][l + 1]$;
      $Avg_d := (iA_i[s] + dS)/(i + 1)$;
      Let l be such that $A_i[l] \leq Avg_d \leq A_i[l + 1]$;
      Let α be such that $Avg_d = \alpha A_i[l] + (1 - \alpha)A_i[l + 1]$;
      $C_d := \alpha C[j][l] + (1 - \alpha)C[j][l + 1]$;
      $V[j][s] := e^{-r\Delta t}(pC_u + (1 - p)C_d)$;
    end for
  end for
  $C = V$;
end for
$C[1][1]$; ▷ C[1][1] is the option price

4. Numerical Experiments

This section gives numerical results of cell averaging when applied to exotic options such as Bermudan option or path-dependent options including Discrete forward starting Asian option and Continuous Asian option. We find that the cell-averaged values are more accurate than other schemes in the sense that these values fall in the interval containing the exact value faster (Bermudan option) or converge to a limiting value faster (path-dependent Asian options). Section 4.1 shows that cell averaging can be used independently to derive a cell averaging tree method. Section 4.2 and Section 4.3 show that cell averaging can be easily combined with other existing tree methods as well. Section 4.2 shows that cell averaged values are so smooth that the Richardson extrapolation can be used further to improve the accuracy. Section 4.3 shows that cell averaging can be extended to price even American options with ease.

4.1. Bermudan option

In this section, we implement the cell averaging method to construct a tree method to price the Bermudan call option with the initial stock price $S_0 = 100$, the risk-free interest rate $r = 0.05$, the dividend $q = 0.1$, the volatility $\sigma = 0.2$, strike price $K = 100$ and the maturity $\tau = 3$. When there are $m = 2$ exercising points, Anderson and Broadie (2004) provided using a binomial method with $n = 2000$ time steps a upper bound of 7.23 and a lower bound of 7.08 for the option price whose exact value is 7.18. See Table 1. See also Anderson and Broadie (2004) for the computational results by Anderson and Broadie and the explanations on them. The proposed cell-averaged values fell into this bound with as low as $n = 200$ time steps. In addition, the bound of Anderson and Broadie has the width of 0.15 with $n = 2000$ time steps while the variation of cell-averaged values for $400 \le n \le 2000$ is only 0.03, which implies that the proposed cell averaging scheme converges fast. Similar results are observed for the Bermudan option with $m = 10$ exercising points and the American option.

n	$m = 2$	$m = 10$	American
50	6.8466	7.5856	7.7360
100	6.9833	7.7896	7.9557
200	7.0855	7.8837	8.0645
400	7.1316	7.9362	8.1199
1000	7.1602	7.9629	8.1532
1500	7.1664	7.9705	8.1605
2000	7.1682	7.9732	8.1642
AB($n = 2000$)	[7.08, 7.23]	[7.81, 8.09]	[7.98, 8.30]
MC-exact	7.18	7.98	8.17

Table 1 : Parameters: The initial stock price $S(0) = 100$, the risk-free interest rate $r = 0.05$, the dividend $q = 0.1$, the volatility $\sigma = 0.2$, strike price $K = 100$ and the maturity $T = 3$ for Bermudan call option and m is the number of discrete exercising points. AB represents the lower and upper bounds for the option price by Anderson and Broadie (2004) using a binomial lattice with $n = 2000$ time steps and $f = 1.06$. f gives the ratio of the critical exercise price under the suboptimal policy to the optimal critical exercise price (See Anderson and Broadie (2004) for details). Anderson and Broadie also presented exact values (MC-exact) in Anderson and Broadie (2004).

4.2. Discrete Asian option

respectively. Since cell averaging reduces oscillations as explained in Section 2.2, the cell averaged values can be later improved by the Richardson extrapolation. Table 2 shows the comparison of several numerical schemes by Večeř, Tavella and Randall (TR), Curran, Hsu and Lyuu (HL) with the cell averaging modification of HL (HL-CA) using the Algorithm 1, and the Richardson extrapolation of HL-CA method (HL-CA*) for the initial price $S_0 = 95, 100, 105$ and various values of the number of monitor points n. The number of time steps $N = nI$. See Hsu and Lyuu (2011) for detailed explanations on those schemes including the parameters used for the simulations.

In Table 2, for $S_0 = 95$, the variations in HL-CA* method from $n = 250$ to $n = 500$ and from $n = 500$ to $n = 1000$ are only 0.002 and 0.001, respectively. On the other hand, corresponding variations in other methods are about 0.02 and 0.01. Similar differences are observed for $S_0 = 100$ or $S_0 = 105$. Thus, the application of cell averaging results in sufficiently smooth convergence and the additional application of the Richardson extrapolation (HL-CA*) produces very rapid convergence. The proposed method is, in this sense, very competitive with many other existing methods.

S_0	n	Večeř	TR	Curran	HL	HL-CA	HL-CA*
95	10	9.2228	9.2149	9.2197	9.2209	8.326652102	10.057893972
	25	8.7080	8.6974	8.7053	8.7137	8.570829831	8.637790724
	50	8.5367	8.5383	8.5340	8.5341	8.504213511	8.437597191
	125	8.4339	8.4304	8.4314	8.4335	8.429317398	8.377522888
	250	8.4001	8.3972	8.3972	8.4003	8.399090474	8.368863550
	500	8.3826	8.3804	8.3801	8.3831	8.382987496	8.366884518
	1000	8.3741	8.3719	8.3715	8.3745	8.374417419	8.365847341
100	10	12.0420	12.0348	12.0390	12.0425	11.128581650	13.015832655
	25	11.4906	11.4803	11.4881	11.4952	11.354548719	11.412606818
	50	11.3068	11.2982	11.3043	11.3079	11.275061820	11.195574920
	125	11.1967	11.1929	11.1940	11.1959	11.192063016	11.134621797
	250	11.1600	11.1573	11.1572	11.1602	11.159164761	11.126266506
	500	11.1416	11.1392	11.1388	11.1417	11.141612382	11.124060003
	1000	11.1322	11.1300	11.1296	11.1325	11.132411787	11.123211192
105	10	15.2234	15.2168	15.2202	15.2225	14.306824430	16.389334632
	25	14.6510	14.6415	14.6483	14.6553	14.521862965	14.555769035
	50	14.4601	14.4519	14.4575	14.4621	14.430821701	14.339780438
	125	14.3455	14.3424	14.3430	14.3493	14.341651015	14.280567994
	250	14.3073	14.3054	14.3048	14.2982	14.306971212	14.272291408
	500	14.2881	14.2866	14.2857	14.2878	14.288557498	14.270143784
	1000	14.2786	14.2771	14.2762	14.2865	14.279030723	14.269503949

Table 2 : Parameters: A discrete forward starting Asian call option is considered with the strike price $K = 100$, the time to maturity $\tau = 1$, the volatility $\sigma = 0.4$, the interest rate $r = 0.1$, and the dividend rate is not considered. Parameters for numerical schemes: intraday period $I = 10$ and the average number of states per node $c = 50$ for Hsu and Lyuu (2011) are considered. The table shows the

comparison of various numerical schemes by Večeř, Tavella and Randall (TR), Curran, Hsu and Lyuu (HL) with the cell averaging modification of HL (HL-CA) using the Algorithm 1, and the Richardson extrapolation of HL-CA method (HL-CA*) for the initial price $S_0 = 95, 100, 105$ and various values of the number of monitor points n (and the number of time steps $N = nI$)

4.3. Continuous Asian option

In this section, we now apply the cell averaging method to value another path-dependent option, a continuous Asian call option, based on the Hull and White (1993) method and then extend it to value an American option.

The initial stock price $S_0 = 50$, strike price $K = 50$, risk-free interest rate $r = 0.1$, volatility $\sigma = 0.3$, expiry date $T = 1$, and dividend rate $q = 0$ for European Asian call option. The parameters for numerical scheme by is $h = 0.001$. Figure 3 shows the errors in the Hull and White binomial method and the cell averaging Hull and White method as the number n of time steps increases, when $\alpha = 0.5$ is used for the Algorithm 2. The solution computed by Monte Carlo simulations based on 10^{-4} time steps and 10^{8} simulation runs is used when the error is measured, which means that Monte Carlo value has statistic error of $O(10^{-4})$. The figure shows that the error from the cell averaged values decreases to zero faster. Table 3 shows the option values from Hull and White binomial method (HW) and the cell averaging HW method (HW-CA) using the Algorithm 2 with $\alpha = 0.5, 0.6, 0.7$, and 0.8 as the number of time steps, n, increases. The values in parenthesis are the errors. We see that the convergence of the cell averaging HW-CA values is faster than that of HW values.

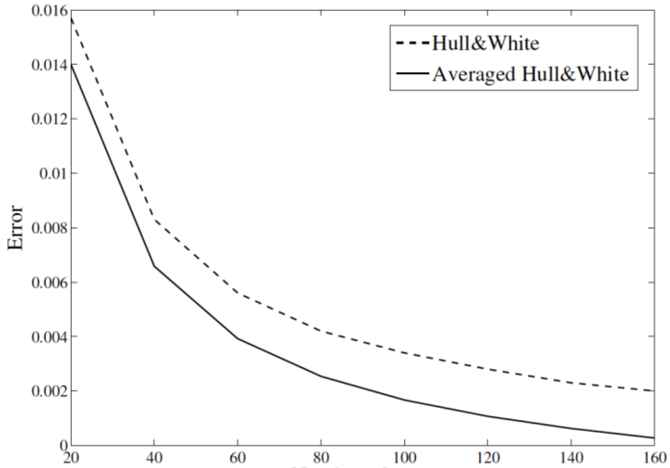

Figure 3 : Parameters: the initial stock price $S_0 = 50$, strike price $K = 50$, risk-free interest rate $r = 0.1$, volatility $\sigma = 0.3$, expiry date $T = 1$, and dividend rate $q = 0$ for European Asian call option. The parameters for numerical scheme by Hull and White (1993) and Algorithm 2 are $h = 0.001$ and $\alpha = 0.5$. The figure shows the errors in option values from Hull and White

binomial method and the cell averaging HW method using the Algorithm 2 as the number of time steps, n, increases.

n	HW	HW-CA			
		$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$
20	4.5120 (0.0157)	4.5137 (0.0140)	4.5145 (0.0132)	4.5153 (0.0124)	4.5163 (0.0114)
40	4.5194 (0.0083)	4.5211 (0.0066)	4.5219 (0.0058)	4.5227 (0.0050)	4.5238 (0.0039)
60	4.5221 (0.0056)	4.5238 (0.0039)	4.5245 (0.0032)	4.5254 (0.0023)	4.5264 (0.0013)
80	4.5235 (0.0042)	4.5252 (0.0025)	4.5259 (0.0018)	4.5268 (0.0009)	4.5278 (0.0001)
			4.5277		

Table 3: Parameters: the initial stock price $S_0 = 50$, strike price $K = 50$, risk-free interest rate $r = 0.1$, volatility $\sigma = 0.3$, expiry date $T = 1$, and dividend rate $q = 0$ for European Asian call option. The parameters for numerical scheme by Hull and White (1993) and Algorithm 2 are $h = 0.001$ and $\alpha = 0.5, 0.6, 0.7, 0.8$. The table shows the option values from Hull and White binomial method (HW) and the cell averaging HW method (HW-CA) using the Algorithm 2 as the number of time steps, n, increases. The values in parenthesis are the errors. The solution computed by Monte Carlo simulations (MC) based on 10^{-4} time steps and 10^{8} simulation runs is used when the errors are measured.

A simple extension of the Algorithm 2 for an exercise of the option results in Table 3.

n	HW	HW-CA			
		$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$
20	4.8127	4.8136	4.8142	4.8149	4.8158
40	4.8881	4.8891	4.8896	4.8902	4.8910
60	4.9174	4.9185	4.9190	4.9196	4.9204
80	4.9336	4.9346	4.9351	4.9357	4.9365

Table 4: Parameters: the initial stock price $S_0 = 50$, strike price $K = 50$, risk-free interest rate $r = 0.1$, volatility $\sigma = 0.3$, expiry date $T = 1$, and dividend rate $q = 0$ for American Asian call option. The parameters for numerical scheme by Hull and White (1993) and Algorithm 2 are $h = 0.001$ and $\alpha = 0.5, 0.6, 0.7, 0.8$. The table shows the option values from Hull and White binomial method (HW) and the cell averaging HW method (HW-CA) using the Algorithm 2 as the number of time steps, n, increases.

The cell averaging method can be easily extended to American options. Table 4 considers an American Asian call option with the same parameters as those for a European Asian call option above using the Hull and White method and the cell averaging Hull and White model.

5. Conclusions

We propose the cell averaging method for pricing the exotic options, in particular path-dependent Asian options. Cell averaging reduces the oscillations of the tree method and thus improves the accuracy. It can be used to derive an independent tree scheme or to be combined with existing methods. It can be even combined with the extrapolation as in Section 4.2 to enhance the accuracy using the fact that the corresponding result is smooth or it can be easily extended to value American path-dependent options as in Section 4.3.

Algorithms 1 and 2 show that the introduction of cell averaging does not increase computational loads much, while numerical experiments validate that cell averaging improves the accuracy of Hsu and Lyuu method and Hull and White method pricing path-dependent Asian options. For instance, cell averaging gives better representative averages than those proposed by Hull and White.

We are currently working on mathematical analysis on the order of convergence of HL-CA and HW-CA methods for path-dependent Asian options.

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